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Equilibrium Points of Nonatomic Games over a Nonreflexive Banach Space*

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Schmeidler's results on the equilibrium points of nonatomic games with strategy sets in Euclidean n-space are generalized to nonatomic games with stategy sets in a separable Banach space whose dual possesses the Radon–Nikodým property.

I. INTRODUCTION

In [10], Schmeidler proved the existence of a Nash equilibrium in games with a nonatomic measure space of players each of whose strategy sets is the unit simplex in Euclidean n-space. For the case where the individual payoff functions are restricted to depend only on the average response of the other players rather than on each of their individual responses, Schmeidler also showed the existence of a pure strategy equilibrium in which almost every player chooses a basic vector in Euclidean n-space as his strategy.

A restrictive aspect of Schmeidler's results is that the number of pure strategies available to each player is uniformly bounded over the set of players. It is natural to ask whether such an assumption can be relaxed. In this paper we generalize the results in [10] to nonatomic games in which each player's strategy set is a weakly compact, convex subset of a separable Banach space whose dual has the Radon–Nikodým property. In this setting, a pure strategy equilibrium is one in which almost every player is limited to a subset of his strategy set consisting only of its extreme points. For equivalent formulations of the hypothesis that a Banach space possess the Radon–Nikodým property, the reader is referred to the comprehensive work of Diestel and Uhl, [3, especially p. 217].

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Schmeidler proved his results through the use of the Fan–Glicksberg fixed point theorem [5, 6] and the theory of integration of set-valued mappings whose range is Euclidean n-space as set out in [1]. Our proofs are modelled after those of Schmeidler but we have to rely on the corresponding theory of Bochner integration of set-valued mappings whose range is a separable Banach space. The fact that such a space is not required to be reflexive gives mathematical depth to the theory.

II. The Model and Results

Let $(T, \mathscr{T}, \mu)$ be a complete, finite, nonatomic measure space, i.e., μ is a real-valued, nonnegative, countably additive, nonatomic measure defined on a complete σ-field $\mathscr{T}$ of subsets of a point set T such that $\mu(T)$ is finite. T is to be interpreted as the set of players.

Let X denote a Banach space over the real numbers R and let X^* be its topological dual. The norm in X and X^* will be denoted by $\|\cdot\|$. $\overline{\text{co}}(A)$, $\text{co}(A)$, $\text{cl}(A)$, $\text{ext}(A)$ will respectively stand for the closed convex hull, convex hull, norm closure and the set of extreme points of A.

Our next set of definitions involve Bochner integrable functions; see [3, Chap. II] for details. Let $L^1(\mu, X)$ denote the space of all (equivalence classes of) X-valued Bochner integrable functions f defined on T with $\|f\| = \int_T \|f(t)\| \, d\mu(t)$. Similarly, $L^\infty(\mu, X)$ denotes the space of all (equivalence classes of) X-valued, Bochner integrable functions defined on T that are essentially bounded, i.e., such that

$$\|f\|_\infty = \text{ess sup}\{\|f(t)\| : t \in T\} < \infty.$$

We shall abbreviate $L^1(\mu, R)$ and $L^\infty(\mu, R)$ to $L^1(\mu)$ and $L^\infty(\mu)$, respectively. A Banach space X has the *Radon–Nikodým property with respect to* $(T, \mathscr{T}, \mu)$ if for each μ-continuous vector measure $G: \mathscr{T} \to X$ of bounded variation, there exists $g \in L^1(\mu, X)$ such that $G(A) = \int_A g(t) \, d\mu(t)$ for all $A \in \mathscr{T}$. X is said to have the *Radon–Nikodým property* (RNP) if X has the Radon–Nikodým property with respect to every finite measure space.

Next, we turn to the measurability of set-valued mappings; see [2, Chap. III]. A set-valued mapping $P: T \to 2^X$ is said to be *measurable* if the graph of P, $G_P = \{(t, x) \in T \times X: x \in P(t)\}$ is an element of $\mathscr{T} \otimes \mathscr{B}(X)$, where $\mathscr{B}(X)$ is the set of Borel subsets of X. P is said to be *integrably bounded* if there exists $g \in L^1(\mu)$ such that $\sup\{\|x\| : x \in P(t)\} \leqslant g(t)$ for almost all t in T. A *selection* from P is a map $\sigma: T \to X$ such that $\sigma(t) \in P(t)$ for almost all t in T. The *integral* of a set-valued mapping P is defined for any $A \in \mathscr{T}$ by

$$\int_A P(t) \, d\mu(t) \equiv \left\{ \int_A \sigma(t) \, d\mu(t) : \sigma \in \mathscr{F}_P \right\}$$

where $\mathscr{F}_p$ denotes the family of all measurable selections from P. For any set-valued mapping P, the set-valued mappings $\overline{\text{co}}\, P$ and $\text{ext}\, P$ will have the obvious meaning. We shall also need the notation $\mathscr{L}_p$ for the set $\{f \in L^1(\mu, X): f(t) \in P(t)$ for almost all t in $T\}$.

A *nonatomic game* $\mathscr{G}$ is a quadruple $[(T, \mathscr{T}, \mu), X, P, u]$, where

1. $(T, \mathscr{T}, \mu)$ is a finite, complete, nonatomic measure space.
2. X is a separable Banach space whose dual possesses RNP.
3. $P: T \to 2^X$ is an integrably bounded map such that $\text{ext}\, P$ is measurable and that for all t in T, $P(t)$ is nonempty, convex and weakly compact.
4. $u: T \times X \times \mathscr{L}_P \to R_+$ is a map such that
   (i) for all $x \in \mathscr{L}_P$, $u(\cdot, \cdot, x)$ is a Borel-measurable function on G_P;
   (ii) for all $t \in T$, $x \in \mathscr{L}_P$, $u(t, \cdot, x)$ is quasi-concave and continuous on $P(t)$;
   (iii) for all $t \in T$, $x \in P(t)$, $u(t, x, \cdot)$ is continuous on $\mathscr{L}_P$.

These conditions on the nonatomic game seem to us natural for our setting; indeed conditions (3) and (4) are suggested by Remark 4 in [10]. We can now state

THEOREM 1. *A nonatomic game $\mathscr{G}$ has an equilibrium strategy, i.e., $x \in \mathscr{L}_P$ such that for almost all t in T*

$$u(t, x(t), x) \geqslant u(t, y, x) \qquad \text{for all} \quad y \in P(t).$$

For our next result we shall need an aggregation and a linearity assumption, i.e.,

ASSUMPTION 1. For all (t, y) in G_P and for all $x \in \mathscr{L}_P$

$$u(t, y, x) = u\left(t, y, \int_T x(t)\, d\mu(t)\right).$$

ASSUMPTION 2. For all $t \in T$, $x \in \mathscr{L}_P$, $u(t, \cdot, x)$ is linear on $P(t)$.

We can now state

THEOREM 2. *Under Assumptions* 1 *and* 2, *a nonatomic game $\mathscr{G}$ has an approximate, pure strategy equilibrium, i.e., $\forall \varepsilon > 0$, there exists $y \in \mathscr{L}_{\text{ext}\, P}$ such that*

$$\left\| u\left(t, y(t), \int_T y(t)\, d\mu(t)\right) - u_t^* \right\| < \varepsilon$$

where

$$u_t^* = \operatorname*{Max}_{z \in P(t)} u\left(t, z, \int_T y(t)\, d\mu(t)\right).$$

III. Proofs

We begin with a

Proof of Theorem 1. The proof is an application of the Fan–Glicksberg fixed point theorem. An equilibrium strategy is a fixed point of the set-valued mapping $\alpha: \mathscr{L}_P \to 2^{\mathscr{L}_P}$, where

$$\alpha(x) = \{y \in \mathscr{L}_P: y(t) \in B(t, x) \text{ for almost all } t \text{ in } T\}$$

$$B(t, x) = \{y(t) \in P(t): u(t, y(t), x) \geqslant u(t, z, x)\ \forall z \in P(t)\}.$$

We have to verify that the mapping α satisfies all the conditions required by the Fan–Glicksberg theorem. We do this in a series of claims.

Claim 1. $\mathscr{L}_P$ is weakly compact.
See proof of Corollary of Theorem 1 in [7].

Claim 2. The graph P belongs to $\mathscr{T} \otimes \mathscr{B}(X)$, where $\mathscr{B}(X)$ is the Borel σ-field of the set X.
We can apply Theorem III.40 and the remark following Definition III.21 in [2], to assert that the graph of $\overline{\text{co}}\,\text{ext}\, P$ belongs to $\mathscr{T} \otimes \mathscr{B}(X)$. Since $P(t)$ is weakly compact and convex for almost all t in T, an application of the Krein–Milman theorem [8, Theorem 11.2.1] completes the proof of the claim.

Claim 3. $\mathscr{L}_P$ is nonempty and convex.
The convexity of $\mathscr{L}_P$ follows directly from the convexity of $P(t)$ for all t in T. The fact that $\mathscr{L}_P$ is nonempty follows, given Claim 2, from the von Neumann–Aumann selection theorem, [2, Theorem III.22]. The fact that such a selection is integrable follows from the fact that P is integrably bounded and that a function f is Bochner integrable if $\int_T \|f(t)\|\, d\mu(t) < \infty$; see [3, p. 45].

Claim 4. For each x in $\mathscr{L}_P$, $\alpha(x)$ is nonempty and convex.
Given quasi-concavity of $u(t, \cdot, x)$, it is clear that $B(t, x)$ is convex and hence $\alpha(x)$ is convex. From the continuity of $u(t, \cdot, x)$ and weak compactness of $P(t)$, we obtain that $B(t, x) \neq \emptyset$ for all t in T. Given Claim 2 and the Borel-measurability of $u(\cdot, \cdot, x)$, we can apply Lemma III.39 in [2] to claim that $B_x: T \to 2^X: B_x(t) \equiv B(t, x)$ has a measurable selection. Integrability follows from the fact that $B(t, x) \subseteq P(t)$.

Claim 5. For all t in T, the graph of $B(t, \cdot)$ is weakly closed in $P(t) \times \mathscr{L}_P$.

This is a straightforward consequence of the continuity of $u(t, \cdot, \cdot)$ on $P(t) \times \mathscr{L}_P$.

Claim 6. The graph of α is weakly closed in $\mathscr{L}_P \times \mathscr{L}_P$.

Let (x_ν, y_ν) converge weakly to (x_0, y_0) such that $y_\nu \in \alpha(x_\nu)$ for all ν. Suppose $y_0 \notin \alpha(x_0)$, i.e., there exists $S \subseteq T$, $\mu(S) > 0$ such that

$$y_0(t) \notin B(t, x_0) \qquad \text{for all} \quad t \in S.$$

Let P_S be the restriction of the mapping P to S and $\alpha_S(x_0) = \{z \in \mathscr{L}_{P_S}: z(t) \in B(t, x_0)\}$. As in our proof of Claim 3, $\alpha_S(x_0)$ is nonempty and convex. It is also closed. This can be seen by considering a sequence $z_\nu \in \alpha_S(x_0)$ which converges to z, i.e., $\lim_{\nu \to \infty} \|z_\nu - z\| = 0$. This implies that for almost all t in T, $\lim_{\nu \to \infty} \|z_\nu(t) - z(t)\| = 0$. Our assertion follows from the closedness of $B(t, x_0)$ which, in turn, is due to the continuity of $u(t, \cdot, x)$ on $P(t)$.

Now let $y_S = \{x \in \mathscr{L}_{P_S}: x(t) = y_0(t)$ for all t in $S\}$. By hypothesis, $y_S \notin \alpha_S(x_0)$. We can now apply the Hahn–Banach theorem (see [4, Theorem 10, p. 417] for the precise form we need) to claim that there exists a nonzero, continuous linear functional, i.e., $f \in (L^1(\mu, X))^*$, $f \neq 0$, such that

$$f(y_s) > f(z) \qquad \text{for all} \quad z \in \alpha_S(x_0). \tag{1}$$

But by the Radon–Nikodým property [3, Theorem 1, p. 98], the dual of $L^1(\mu, X)$ is $L^\infty(\mu, X^*)$. We can thus represent f by $g \in L^\infty(\mu, X^*)$ and rewrite (1) as

$$\int_S \langle y_0(t), g(t) \rangle \, d\mu(t) > \int_S \langle x(t), g(t) \rangle \, d\mu(t) \qquad \forall x \in \alpha_S(x_0) \tag{2}$$

Since y_ν converges weakly to y_0, certainly,

$$\lim_{\nu \to \infty} \int_S \langle y_\nu(t), g(t) \rangle \, d\mu(t) = \int_S \langle y_0(t), g(t) \rangle \, d\mu(t) \tag{3}$$

Since $|\langle y_\nu(t), g(t) \rangle| \leqslant \|y_\nu(t)\| \cdot \|g\|_\infty$, $y_\nu(t) \in P(t)$ and P is integrably bounded, we can apply Fatou's lemma [4, III.9.35] to conclude

$$\lim_{\nu \to \infty} \int_S \langle y_\nu(t), g(t) \rangle \, d\mu(t) \leqslant \int_S \limsup_{\nu \to \infty} \langle y_\nu(t), g(t) \rangle \, d\mu(t). \tag{4}$$

Now $r(t) \equiv \limsup_{\nu \to \infty} \langle y_\nu(t), g(t) \rangle = \lim_{k \to \infty} \langle y_{\nu_k}(t), g(t) \rangle$. Since $y_{\nu_k}(t) \in P(t)$ and $P(t)$ is weakly compact, the Eberlein–Smulian theorem [4, V.6.1] guarantees that $r(t)$ is attained at $\bar{y}(t) \equiv \lim_{j \to \infty} y^j_{\nu_k}(t)$. But by Claim 5, $\bar{y}(t) \in B(t, x_0)$ and we have a contradiction to (2).

We can now apply the Fan–Glicksberg theorem to complete the proof. Q.E.D.

Proof of Theorem 2. By an appeal to Theorem 1, we know that there exists $x \in \mathscr{L}_P$ such that for almost all t, $x(t) \in B(t, x_T)$, where $x_T \equiv \int_T x(t)\, d\mu(t)$. Let

$$B^e(t, x_T) = \{x \in \operatorname{ext} P(t)\colon u(t, x(t), x_T) \geqslant u(t, y, x_T) \qquad \forall y \in \operatorname{ext} P(t)\}.$$

We can assert

Claim 1. For almost all t in T, $B(t, x_T) = \overline{\text{co}}\, B^e(t, x_T)$.

Since $P(t)$ is a nonempty, weakly compact and convex subset of X, we can apply a corollary of the Krein–Milman theorem [8, Corollary 11.2.3] to assert that there exists $p(t) \in \operatorname{ext} P(t)$ such that

$$u(t, p(t), x_T) = m_t^* = \sup_{q \in P(t)} u(t, q, x_T).$$

Since $p(t) \in B^e(t, x_T)$, it is clear that $B^e(t, x_T) \subseteq B(t, x_T)$. Since $B(t, x_T)$ is closed and convex, $\overline{\text{co}}\, B^e(t, x_T) \subseteq B(t, x_T)$. Now suppose the containment is strict i.e. there exists $z \in B(t, x_T)$ and $z \notin \overline{\text{co}}\, B^e(t, x_T)$. By the Krein–Milman theorem [8, Theorem 11.2.1] there exist z_1, z_2 in $\operatorname{ext} P(t)$ such that $z = z_1 + (1 + \lambda) z_2$ for some $0 < \lambda < 1$ with at least one $z_i \notin B^e(t, x_T)$. But then $u(t, z, x_T) < m^*$, a contradiction.

Claim 2. $\int_T B(t, x_T)\, d\mu(t) = \text{cl} \int_T B^e(t, x_T)\, d\mu(t)$.

Since $\operatorname{ext} P$ and $u(\cdot)$ are measurable, we can appeal to the application of Lemma III.39 in [2] to assert that $B^e(\cdot, x_T)$ is a measurable, set-valued mapping. It is certainly integrably bounded such that for almost all t in T, $B^e(t, x_T)$ is nonempty and that $\overline{\text{co}}\, B^e(t, x_T)$ is weakly compact. The latter property follows from Claim 1 and the fact that $B(t, x_T)$ is a closed, convex and hence weakly closed, subset of a weakly compact set [4, I.5.7a]. Since X^* has the Radon–Nikodým property, we can now apply Theorem 2′ along with the Remark in [7] to complete the proof of the Claim.

Now from Claim 2, we can assert that for all $\varepsilon > 0$, there exist for almost all t, $y(t) \in B^e(t, x_T)$ such that $\|y_T - x_T\| < \varepsilon$. This implies, by continuity of $u(t, x(t), \cdot)$, that for all $\varepsilon > 0$, and for all t in T,

$$\|u(t, x(t), x_T) - u(t, x(t), y_T)\| < \varepsilon$$

But from Claim 1, for almost all t in T,

$$u(t, y, x_T) = u(t, x(t), x_T)$$

and the proof is finished. Q.E.D.

IV. Concluding Remarks

1. The question as to whether Theorem 2 is valid without the aggregation Assumption 1, has already been answered negatively by Schmeidler, [10, Remark 2]. However, the question remains as to whether we can do without the linearity Assumption 2. The difficulty is in the proof of Claim 1; in particular in showing that a quasi-concave function attains its maximum at an extreme point. It is not clear to us now this difficulty is overcome even in the set-up of [10], where it reduces to showing the non-emptiness of $\{e_i \mid e_i \in B(t, \hat{x})\}$, (equation c, p. 298 in [10]).

2. It is clear that relaxation of the separability assumption will require fundamentally new mathematical techniques. In particular, the St. Beuve extension of the Aumann measurable selection theorem requires the underlying space to be Souslin, see [2, Theorem III.22].

3. We have used the Radon–Nikodým property of X^* in working with $L^\infty(\mu, X^*)$ as the dual of $L^1(\mu, X)$. This is required to establish Claims 1 and 6 in the proof of Theorem 1 and Claim 2 in the proof of Theorem 2. The question as to whether one or both of our results are false without it remains open.

4. The key to the approximation in Theorem 2 is the fact that Lyapunov's theorem does not generalize to infinite spaces; see [3, Chap. IX.1]. In this connection, we should draw the reader's attention to a similar approximate theorem for a setting with a finite number of agents each of whom has strategy sets in Euclidean n-space; see Rashid [9].

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